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New bounds of arboricity of graph

B. Nikfarjam[1*], M. Yunusi[2] and M. Yahaghi[3]

1- PHD candidate tajik national university
2- Head of math tajik national university
3- Islamic Azad university of Ramsar, Iran

Corresponding author: B. Nikfarjam

ABSTRACT: let G be a graph with n vertices and m edge and γ, θ, θ_0 denote, respectively , the arboricity, thickness and outer thickness of G . we establish inequalities; each of which is best possible up to a constant, between pairs of these parameters. In particular, we show.
$\gamma(G) \leq 4\theta(G)$
$\gamma(G) \leq {}^{3}/_{2}\,\theta_0(G)$
$\gamma(G) \leq \sqrt{m} + O(1)$
$\gamma(G) \leq \frac{1}{2}\sqrt{m} + O(1)$

Keywords**:** arboricity.

INTRODUCTION

There are several different ways to characterize the embeddability of a graph G. the thickness $\theta(G)$ of a graph is the minimum number of planar sub graph into which the graph can be decomposed. The thickness problem , asking for the thickness of a given graph G, is NP-hard so there is little hope to find a polynomial time algorithm for the thickness problem on general graphs. The thickness problem has application is VLSI design.[G.chartland and et al] the outer thickness $\theta_0(G)$ of a graph G is minimum number of outer planar sub graph into which the graph can be decomposed . The arboricity $\gamma(G)$ of a graph G is the minimum number of forests whose union in G. is the minimum number of forests whose union in G .

Theorectical bounds

In this section we give some bounds for the thickness and outer thickness of a graph.

Theorem (2.1) [F.harary]. Let G'=(V,E') be a maximum planar sub graph of graph G=(V,E). then $|E'| \leq 3|V| - 6$.
Theorem (2.2) [F.harary] let G'=(V,E') be a maximum planar sub graph of a graph G=(V,E) which does not contain any triangle . then $|E'| \leq 2|V| - 4$.

Theorem (2.3) [K.S.Kedlaya] let G'=(V,E') be a maximum outer planar sub graph of a graph $G=(V,E)$ then $|E'| \leq 2|V| - 3$.
Theorem (2.4) [K.S.Kedlaya] let G'=(V,E') be a maximum outer planer sub graph of a graph G=(V,E) which does not .contain any triangle . then
$|E'| \leq {}^{3|V|}/_{2} - 2.$
Def (2.1) the thickness of a graph, denoted by $\theta(G)$, is the minimum of planar sub graph in to which the graph can be decomposed. Evidently, $\theta(G) = 1$ if and only if G is planar.
Theorem (2.1) if G=(V,E) is a graph with ($|V| = n > 2$) and $|E| = m,$ then :

i) $\theta(G) \geq \left\lceil \frac{m}{3n-6} \right\rceil$

ii) $\theta(G) \geq \left\lceil \frac{m}{2n-4} \right\rceil$, if G has no triangle.

Proof. by theorem (2.1) , the denominator is the maximum size of each planar sub graph. The pigenhole principle then yields the inequality.

Theorem (2.2) [C.st.j.Nash-Williams] if G=(V,E) is a graph with ($|V| = n > 10$) and $|E| = m$ maximal degree d, then

i) $\theta(k_n) \leq \left\lfloor \frac{n+7}{6} \right\rfloor$

ii) $\theta(G) \leq \left\lfloor \sqrt{\frac{m}{3}} + \frac{7}{6} \right\rfloor$

iii) $\theta(G) \geq \left\lceil \frac{d}{2} \right\rceil$

Theorem (2.3) [Beink , L and et al] . The thickness of the complete bipartite graph $k_{m,n}$ is $\theta(k_{m,n}) = \left\lfloor \frac{m.n}{2(m+n-2)} \right\rfloor$ except if m and n are both odd , $m \leq n$ and there is an integer k satisfying $n = \left\lfloor \frac{2k(m-2)}{m-2k} \right\rfloor$

Theorem (2.4) [Beink , L and et al] . the thickness of the complete bipartite graph $k_{n,n}$ is

$\theta(K_{n,n}) = \left\lfloor \frac{n+5}{4} \right\rfloor$.

Theorem (2.5) [A.M.dean and et al] let G be a graph with minimum degree δ and maximum degree Δ . then $\left\lceil \frac{\delta+1}{4} \right\rceil \leq \theta(G) \leq \left\lceil \frac{\Delta}{2} \right\rceil$.

Theorem (2.6) [Kleiner and et al] the thickness of the hypercube Q_n is

$\theta(Q_n) = \left\lceil \frac{n+1}{4} \right\rceil$

New bounds of arboricity

It is well-known that thickness , outer thickness, arborcity re within a constant factor of each other . in particular conclave recently proved a longstanding conjecture that every planar graph G has outer thickness $\theta(G) \leq 2$.
Thus $\theta_0(G) \leq 2\,\theta(G)$.
Also health has shown that a planar graph can be divided into two outer planar graphs . There fore $\theta_0(G) \leq 2\,\theta(G)$
Nash-Williams gave the exact solution for arboricty.

Theorem (3.1) [C.st.j.Nash-Williams] (Nash-Williams)
Let G be a graph . then $\gamma = max \left\lceil \frac{m_H}{n_H-1} \right\rceil$
Where the maximum is taken over all nontrivial subgraph H of G.

Claim(3.1) if G be a graph then $\gamma(G) \leq 3\theta G)$

Proof. according to theorem (2.4) $m \leq {}^{3n}/_{2} - 2$

$$m \leq {}^{3n}/_{2} - 2 =< \frac{3n-4}{2} \rightarrow m \leq \frac{3n-4}{2} \leq \frac{3n-3}{2} = \frac{3(n-1)}{2}$$

$$\xrightarrow{theorem\ (3.1)} \gamma(G) \leq {}^{3}/_{2}\,\theta_0(G).$$

By the way since $\theta_0(G) \leq 2\,\theta(G)$ then
$\gamma\,(G) \leq {}^{3}/_{2} \times 2\theta\,(G) = 3\theta(G)$

Claim (3.2) every planar graph G satisfies $m \leq 2(n-1)$ .thus $\gamma\,(G) \leq 2\theta_0(G)$

Proof . according to theorem (2.3) $m \leq 2n-3$

$$m \leq 2n-3 \leq 2n-2 \rightarrow m \leq 2(n-1) \xrightarrow{theorem\ (3.1)} \gamma(G) \leq 2\,\theta_0(G)$$

relation between thickness and arboricity [B.Nikfarjam and et al]

Now we show that new results for arboricity of a graph.

Claim (4.1) . if G be a graph then

i. $\gamma(G) \leq 2\theta G)$

ii. $\gamma(G) \leq 4\theta G)$

Proof. According to theorem (2.2) $m \leq 2n-4$

$$m \leq 2n-4 \leq 2n-2 = 2(n-1) \rightarrow \mathrm{m} \leq 2(\mathrm{n}-1) \xrightarrow{theorem\ (3.1)} \gamma(G) \leq 2\,\theta(G)$$

also since $\theta\,(G) \leq \theta_0(G)$ and by claim (3.2) $\theta_0(G) \leq 2\,\theta(G)$ then
$\gamma\,(G) \leq 2\,\theta_0\,(G) \leq 2 \times 2\,\theta(G) = 4\,\theta(G)$.

But by applying Nash-Williams , results , dean at al showed that $\gamma\,(G) \leq \left\lceil \sqrt{\frac{m}{2}} \right\rceil$.
This gives also a lower bound for outer thickness . the upper bound is of the right order , since the outer thickness of the complete graph with n vertices is o(n) . on the other hand , since $\theta_0\,(\,k_n\,)$ is approximately $\sqrt{\frac{m}{8}}$ and $\theta_0\,(\,k_{n,n}\,)$ is approximately $\sqrt{\frac{m}{8}}$, it seems that the constant is not the best possible . also conjecture the following upper bound for outer thickness. [T.Poranen]

Conjecture (4.1) $\theta_0(G) \leq \sqrt{\frac{m}{8}} + o\,(1)$ for an arbitrary graph G with m edges.

Theorem (4.1) [A.M.dean and et al] $\theta(G) \leq \sqrt{\frac{m}{16}} + o\,(1)$ for an arbitrary graph G with m edges.

Claim (4.2) $\gamma\,(G) \leq \sqrt{m} + o\,(1)$ for an arbitrary graph G with m edges.

Proof. According to theorem (4.1) since $\theta(G) \leq \sqrt{\frac{m}{16}} + o\,(1)$

$$4\,\theta(G) \leq 4\sqrt{\frac{m}{16}} + 4\,o\,(1)$$
$$4\,\theta(G) \leq \sqrt{m} + o\,(1)$$

But by claim (4.1) $\gamma\,(G) \leq 4\,\theta\,(G)$ then $\gamma(G) \leq \sqrt{m} + o\,(1)$.

Claim (4.3) $\gamma(G) \leq \frac{1}{2}\sqrt{m} + o\,(1)$ for an arbitrary graph G with m edges.

Proof. According to theorem (4.1) since $\theta(G) \leq \sqrt{\frac{m}{16}} + o\,(1)$

$$2\,\theta(G) \leq 2\sqrt{\frac{m}{16}} + 2\,o\,(1) \rightarrow 2\,\theta(G) \leq \frac{1}{2}\sqrt{m} + o\,(1).$$

But by claim (4.1)) $\gamma\,(G) \leq 2\,\theta\,(G)$ then $\gamma(G) \leq \frac{1}{2}\sqrt{m} + o\,(1)$.

CONCULSION

In this paper we present some results have concerning the thickness and edge of a graph. In particular, bounds on the arboricity of graph are given. It seems that bounds are not unique.

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